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An Improved Estimate of the Degree of Monotone Interpolation

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1. INTRODUCTION

Let $0 = x_0 < x_1 < \cdots < x_k = 1$ and let $y_0, y_1, \ldots, y_k$ be real numbers such that $y_{j-1} \neq y_j$, $j = 1, 2, \ldots, k$. It is a result of Wolibner [7], Kammerer [2], and Young [8] that there exists an algebraic polynomial $p(x)$ such that:

(i) $p(x_j) = y_j$, $j = 0, 1, \ldots, k$, and

(ii) $p(x)$ is increasing on $I_j = (x_{j-1}, x_j)$ if $y_j > y_{j-1}$ and decreasing on I_j if $y_j < y_{j-1}$, $j = 1, 2, \ldots, k$.

A polynomial with properties (i) and (ii) is said to interpolate *piecewise monotonely*; in case $y_j > y_{j-1}$ for all j (or if $y_j < y_{j-1}$ for all j), $p(x)$ is said to interpolate *monotonely*. The smallest degree of a polynomial that interpolates the values $Y = \{y_0, y_1, \ldots, y_k\}$ at the points $X = \{x_0, x_1, \ldots, x_k\}$ (piecewise) monotonely is called the *degree of (piecewise) monotone* interpolation of Y *with respect to* X, and is denoted by $N(X; Y)$. Passow and Raymon [6] have obtained bounds on $N(X; Y)$. In this note we obtain improved estimates for the case of monotone interpolation.

2. THE MAIN RESULT

Let $\Delta = \Delta(Y) = \min_{1 \leqslant j \leqslant k} | y_j - y_{j-1} |$, $\alpha = \alpha(X) = \min_{1 \leqslant j \leqslant k} (x_j - x_{j-1})$, $M = M(X; Y) = \max_{1 \leqslant j \leqslant k} |(y_j - y_{j-1})/(x_j - x_{j-1})|$, P_n = the set of all algebraic polynomials of degree $\leqslant n$, and $E_n(f) = \inf\{\| f - p \|_\infty, p \in P_n\}$.

For f monotone increasing, let $E_n^*(f) = \inf\{\| f - p \|_\infty, p \in P_n, p'(x) \geqslant 0$ on $[0, 1]\}$.

In [6] it was shown that if $y_0 < y_1 < \cdots < y_k$, then there exists a constant A_0 such that $N(X; Y) \leqslant A_0(M/\Delta)$. Our improved estimate is as follows.

THEOREM. *Let* $y_0 < y_1 < \cdots < y_k$. *Then there exist constants,* A_r, $r = 0, 1, 2, \ldots$, *such that*

$$N(X; Y) \leqslant \inf_r A_r(M/\alpha^r \Delta)^{1/(r+1)}.$$

Remark 1. The constants in the Theorem satisfy $A_0 \leqslant A_1 \leqslant A_2 \leqslant \cdots$. On the other hand,

$$(M/\alpha^r \Delta)^{1/(r+1)} \leqslant (M/\alpha^{r-1}\Delta)^{1/r} \qquad \text{for any configuration.}$$

Remark 2. The proof in [6] was based on the degree of monotone approximation of piecewise linear functions, using estimates of Lorentz and Zeller [4]. They, individually, generalized this result in [3, 9], and we apply these extensions to the monotone approximation of splines with specified deficiency.

LEMMA 1 [3, p. 209]. *If* $f \in C^1[0, 1]$ *is increasing, then* $E_n^*(f) \leqslant (C/n)\, \omega(f'; 1/n)$, *where* ω *is the modulus of continuity of* f *and* C *is an absolute constant.*

LEMMA 2 [9, p. 524]. *Let* f *be increasing on* $[0, 1]$ *and assume that* $E_n(f) = O(n^{-3})$. *Suppose that* f' *has a finite number of zeros in* $[0, 1]$ *and that* $E_{n-1}(f') = O(nE_n(f))$. *Then* $E_n^*(f) = O(E_n(f))$.

Proof of Theorem. Let $\epsilon = \Delta/4$ and let T be the set of the 2^{k+1} sets of points $\{y_0 \pm \epsilon, y_1 \pm \epsilon, \ldots, y_k \pm \epsilon\}$, where the choices of $\pm$ are made independently. We enumerate the sets in T and denote the ith set by $S_i = \{z_0^{(i)}, z_1^{(i)}, \ldots, z_k^{(i)}\}$. Note that our choice of ϵ guarantees that $z_{j-1}^{(i)} < z_j^{(i)}$, $i = 1, 2, \ldots, 2^{k+1}$, $j = 1, 2, \ldots, k$. For each i we now construct a spline f_i of degree $2r + 1$, with deficiency $r + 1$, having the following properties:

(a) $f_i(x_j) = z_j^{(i)}, j = 0, 1, \ldots, k$;

(b) $f_i^{(s)}(x_j) = 0, j = 0, 1, \ldots, k, s = 1, 2, \ldots, r$;

(c) $f_i \in P_{2r+1}$ on each subinterval $[x_{j-1}, x_j]$, $j = 1, 2, \ldots, k$;

(d) $f_i \in C^r[0, 1]$.

The existence of f_i was proved in [5], and it was also shown there that f_i is monotone. We will now show that $f_i^{(r)} \in \mathrm{Lip}_A\, 1$, where $A = B_r M/\alpha^r$, B_r a constant. The three following results are immediate.

LEMMA 3. *Let* $q(x) = a_1 \int_0^x t^r(1 - t)^r\, dt + y_1$, *where* $a_1 = (y_2 - y_1)/\int_0^1 t^r(1 - t)^r\, dt$. *Then* q *is the unique polynomial in* P_{2r+1} *satisfying* $q(0) = y_1$, $q(1) = y_2$, $q^{(s)}(0) = q^{(s)}(1) = 0$, $s = 1, 2, \ldots, r$.

COROLLARY. *$p(x) = q((x - a)/(b - a))$ is the unique polynomial in P_{2r+1} satisfying $p(a) = y_1$, $p(b) = y_2$, $p^{(s)}(a) = p^{(s)}(b) = 0$, $s = 1, 2,..., r$.*

LEMMA 4. *Let $M_r = \max_{0 \leqslant x \leqslant 1} |(d^{r+1}/dx^{r+1}) \int_0^x t^r(1-t)^r\, dt|$, and let p be as in the corollary. Then,*

$$\max_{a \leqslant x \leqslant b} |p^{(r+1)}(x)| = (a_1 M_r/(b-a)^{r+1}) = ((y_2 - y_1) M_r'/(b-a)^{r+1}),$$

where $M_r' = M_r / \int_0^1 t^r(1-t)^r\, dt$.

Now $f_i(x_j) = z_j^{(i)}$, and $z_j^{(i)} - z_{j-1}^{(i)} \leqslant (y_j + \epsilon) - (y_{j-1} - \epsilon) = (y_j - y_{j-1}) + 2\epsilon \leqslant (3/2)(y_j - y_{j-1})$. Thus, by Lemma 4,

$$\begin{aligned} \max_{x_{j-1} \leqslant x \leqslant x_j} |f_i^{(r+1)}(x)| &\leqslant \frac{3}{2} \frac{(y_j - y_{j-1}) M_r'}{(x_j - x_{j-1})^{r+1}} \\ &= \frac{3}{2} \left(\frac{y_j - y_{j-1}}{x_j - x_{j-1}} \right) \frac{M_r'}{(x_j - x_{j-1})^r} \\ &\leqslant \frac{3}{2} \frac{MM_r'}{\alpha^r} = \frac{B_r M}{\alpha^r}. \end{aligned}$$

The bound is independent of j. Hence, $f_i^{(r)} \in \mathrm{Lip}_A\, 1$, where $A = B_r M/\alpha^r$.

Case 1. $r = 1$. By Lemma 1, $E_n^*(f_i) \leqslant CB_1 M/\alpha n^2$. We now approximate each f_i to within $\Delta/8 = \epsilon/2$ by a monotone polynomial. By Lemma 1 this can be accomplished by a polynomial q_i whose degree does not exceed $(8CB_1 M/\alpha\Delta)^{1/2} = A_1(M/\alpha\Delta)^{1/2}$. The vector $(y_0, y_1, ..., y_k)$ is thus contained in the convex hull of the vectors $(q_i(x_0), q_i(x_1), ..., q_i(x_k))$, $i = 1, 2, ..., 2^{k+1}$. Hence there exists a convex linear combination of the q_i's which gives rise to a polynomial p which interpolates Y monotonely. Since the degree of each $q_i \leqslant A_1(M/\alpha\Delta)^{1/2}$, we have $N(X; Y) \leqslant A_1(M/\alpha\Delta)^{1/2}$.

Case 2. $r \geqslant 2$. Note that f_i' has a finite number of zeros in $[0, 1]$. Moreover, since $f_i \in C^r[0, 1]$ and $f_i^{(r+1)}$ has jump discontinuities in $[0, 1]$, $E_n(f_i)$ has exact order of magnitude $1/n^{r+1}$, while $E_{n-1}(f_i')$ has exact order of magnitude $1/n^r$ [1, p. 14]. Thus $E_{n-1}(f_i) = O(nE_n(f_i))$, so that f_i satisfies the conditions of Lemma 2. Hence $E_n^*(f_i) = O(E_n(f_i)) \leqslant (C_r B_r M/\alpha^r n^{r+1})$ [1, p. 13]. The proof now proceeds as in Case 1, and we obtain $N(X; Y) \leqslant A_r(M/\alpha^r\Delta)^{1/(r+1)}$.

EXAMPLE. Consider monotone interpolation of $f(x) = x^{1/2}$ on $[0, 1]$ at equally spaced nodes, $x_j = j/k$, $j = 0, 1, ..., k$. Then $M = k^{1/2}$, $\alpha = 1/k$ and $\Delta \geqslant 1/2k$. Thus the result of [6] yields $N(X; Y) \leqslant A_0(M/\Delta) \leqslant 2A_0 k^{3/2}$. We obtain $N(X; Y) \leqslant \inf_{r=0,1,2,...} A_r[2k^{(r+(3/2))}]^{1/(r+1)}$.

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